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ON THE COMPLEX GEOMETRY OF A CLASS OF NON-KÄHLERIAN MANIFOLDS

BY

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ABSTRACT

In a recent paper a class of complex, compact and non-Kählerian manifolds was constructed by S. López de Medrano and A. Verjowsky. This class contains as particular cases Calabi–Eckmann manifolds, almost all Hopf manifolds and many of the examples given previously by J.-J. Loeb and M. Nicolau. In this paper we show that these manifolds are endowed with a natural non-singular vector field which is transversely Kählerian, and that analytic subsets of appropriate dimensions are tangent to this vector field. This permits to give a precise description of these sets in the generic case. In the proof, an important role is played by some complex abelian groups which are biholomorphic to big domains in these manifolds.

0. Introduction

In [2] we presented a general procedure of construction of complex structures on a product of two odd-dimensional spheres, giving in particular all the previously known examples: elliptic curves, Hopf manifolds and Calabi–Eckmann manifolds. These complex manifolds were obtained as a part of the orbit space of a holomorphic vector field in $\mathbb{C}^n$ of Poincaré–Dulac type. In [3], using vector

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fields of Siegel type, S. López de Medrano and A. Verjovsky built a large class of compact complex manifolds (called here LM-V manifolds) which includes almost all our examples.

Except in the case of elliptic curves, LM-V manifolds do not admit a Kählerian structure. Nevertheless they enjoy a very special property: we show that on any such manifold there exists a naturally defined holomorphic vector field η which is non-singular and transversely Kählerian. This means that the foliation defined by η is transversely modeled on a Kählerian manifold. This generalizes the fact that classical Hopf manifolds and Calabi–Eckmann manifolds are principal bundles over Kählerian manifolds (namely $\mathbb{P}^m$ and $\mathbb{P}^r \times \mathbb{P}^s$) with fibre an elliptic curve. In these cases η is just the fundamental vector field of the action. It is also well known that in the case of classical Hopf manifolds and Calabi–Eckmann manifolds any complex submanifold Y of positive dimension is saturated by the flow of η and therefore Y is the pullback of a complex submanifold in the base space of the fibration. In Theorem 1 we generalize this result to LM-V manifolds. The proof makes an essential use of the fact that the flow η is transversely Kählerian. Using also some properties of complex abelian groups we give in Theorem 2 a description of the complex submanifolds of LM-V manifolds in many cases.

Given a real manifold M we shall denote by TM its tangent bundle and by $(TM)_x$ the tangent space at $x \in M$. If M is a complex manifold, J will denote the tensor field defining the complex structure. Then the real vector bundle TM inherits a complex structure defined by $i \cdot v = Jv$ and we set $\mathbb{C}v = \mathbb{R}v \oplus \mathbb{R}Jv$. Through the paper any holomorphic vector field η on a complex manifold M will be considered as a real vector field. With this convention, if a holomorphic vector field η is tangent to a complex submanifold Y of M at a point $z \in Y$ then $\mathbb{C}\eta_z \subset (TY)_z$.

1. Transversely Hermitian vector fields

Definition 1: Let M be a complex manifold and v a (real) non-singular vector field on M. A (real) 2-form ω is said to be *transversely Hermitian* with respect to v if

(1) $J\omega = \omega$ and, for any $z \in M$, $\operatorname{Ker} \omega_z = \mathbb{C}v_z = \mathbb{R}v_z \oplus \mathbb{R}Jv_z$, and
(2) the Hermitian quadratic form h on $TM/\mathbb{C}v$ induced by ω is positive definite (recall that h is given by $h(u_1, u_2) = \omega(Ju_1, u_2) + i\omega(u_1, u_2)$).

If the 2-form ω is closed we say that v is transversely Kählerian.

The following proposition is essentially due to Abe [1].

PROPOSITION 1: *Let M, v and ω be as in the above definition and suppose that M is compact and ω^k is exact for some $k \in \mathbb{N}^*$. Then any analytic subset Y of M of pure dimension k is tangent to $\mathbb{C}v$ (at the smooth points of Y).*

Remark: We will use the above proposition mainly when the form ω is closed. In this situation the vector fields v and iv define a 2-dimensional foliation and the form ω is **basic** with respect it, i.e. ω is left invariant by the flows of v and iv. Moreover, in this case $\omega^{k'}$ is also exact for any $k' \geq k$ and therefore the conclusion of the proposition also holds for any analytic subset Y of pure dimension $\geq k$.

Proof: Assume that Y has pure dimension k. There is a canonical orientation on Y given by the complex structure J. Namely, if $\{e_1, \ldots, e_k\}$ is a $\mathbb{C}$-basis of $(TY)_y$ (at a smooth point $y \in Y$) then $\{e_1, Je_1, \ldots, e_k, Je_k\}$ is positive. Let ω_y denote the restriction of ω to $(TY)_y$. One sees easily that either $\mathbb{C}v$ is contained in $(TY)_y$, and then $\omega_y^k = 0$, or $\omega_y^k > 0$. Using Stokes formula (for singular analytic sets) and the exactness of ω^k one obtains

$$\int_Y \omega^k = 0.$$

But the positivity of ω^k implies then that $\omega^k \equiv 0$ and therefore $\mathbb{C}v$ is tangent to Y at every smooth point. ∎

2. The manifolds $N = N(\Lambda)$

Let $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ be a sequence of non-zero complex numbers fulfilling the following two conditions:

(1) Λ is in the Siegel domain, i.e. the convex hull of $\lambda_1, \ldots, \lambda_n$ contains 0,

(2) one has $0 \notin [\lambda_i, \lambda_j]$ for $i \neq j$.

In [3] S. López de Medrano and A. Verjovsky associate, to any such sequence Λ, a compact complex manifold $N = N(\Lambda)$ (called a LM-V manifold for short), which can be constructed as follows. Let us consider the holomorphic vector field on $\mathbb{C}^n$ defined by

$$\xi = \sum_{j=1}^{n} \lambda_j z_j \frac{\partial}{\partial z_j}.$$

It induces a $\mathbb{C}$-action on $\mathbb{C}^n$ given by

$$(t, z) \in \mathbb{C} \times \mathbb{C}^n \to (e^{\lambda_1 t}, \ldots, e^{\lambda_n t}) \in \mathbb{C}^n.$$

For a given point $z \in \mathbb{C}^n \setminus \{0\}$ let I_z denote the subset of $\{1, \ldots, n\}$ characterized by $j \in I_z \Leftrightarrow z_j \neq 0$ and define

$$\mathcal{S} = \{z \in \mathbb{C}^n \setminus \{0\} | \text{ the convex hull of the } (\lambda_j)_{j \in I_z} \text{ contains } 0\}.$$

The set $\mathcal{S}$ is open in $\mathbb{C}^n$ and can also be described as the union of all the Siegel leaves of ξ, i.e. orbits of ξ which do not accumulate to the origin.

The radial vector field

$$R = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}$$

induces a $\mathbb{C}^*$-action given by the non-zero homotheties of $\mathbb{C}^n$. Since the vector fields ξ and R commute they define a $\mathbb{C} \times \mathbb{C}^*$-action on $\mathbb{C}^n$ which restricts to $\mathcal{S}$. It is shown in [3] that the quotient $\mathcal{S}/\mathbb{C} \times \mathbb{C}^*$ is a compact complex manifold denoted by N. We shall also need the following alternative description of N (cf. [3]). Define

$$M' = \{z \in \mathbb{C}^n \mid \sum \lambda_j |z_j|^2 = 0\}.$$

Every Siegel leaf intersects M' at a unique point and the natural map

$$M' \to \mathcal{S}/\mathbb{C}^*$$

is a diffeomorphism of real manifolds. The intersection $M = M' \cap S^{2n-1}$ is a real manifold with a S^1-action induced by the homotheties of modulus 1, that is, by the flow associated to the vector field iR. Moreover, there is a commutative diagram

(D)

$$\begin{array}{ccc} M & & \\ \pi \downarrow & \overset{\psi}{\searrow} & \\ M/S^1 & \xrightarrow[\phi]{} & N = \mathcal{S}/\mathbb{C} \times \mathbb{C}^* \end{array}$$

where π is the canonical projection, ψ is the composition map of the natural inclusion $M \hookrightarrow \mathcal{S}$ and the projection $\mathcal{S} \to \mathcal{S}/\mathbb{C} \times \mathbb{C}^*$ and ϕ is a real analytic diffeomorphism between M/S^1 and N.

Using this fact, López de Medrano and Verjovsky have studied in [3] the topology of N. In particular they have proved that for $n > 3$ there is no Kählerian structure on N (in fact there is no symplectic structure on it). In contrast we shall build a transversely Kählerian holomorphic vector field η on N, i.e. a non-singular holomorphic vector field η with the property that there exists a 2-form ω which is transversely Hermitian with respect to η and closed.

The special case where $\Lambda = (\lambda_1, \dots, \lambda_n)$ with $\operatorname{Im}\lambda_j > 0$ for $j \neq n$ and $\operatorname{Im}\lambda_n < 0$ was studied by the authors in [2]. In this situation N is diffeomorphic to the product of two odd dimensional spheres.

3. A transversely Kählerian vector field on N

In this section we construct a non-singular vector field η on N and a differentiable 2-form ω which is closed and transversely Hermitian with respect to η in the sense of Definition 1.

The holomorphic vector field $\tilde{\eta}$ on $\mathbb{C}^n$ defined by

$$\tilde{\eta} = \sum_{j=1}^{n} (\operatorname{Re}\lambda_j) z_j \frac{\partial}{\partial z_j}$$

commutes with ξ and R. Therefore it projects onto a holomorphic vector field η on $N = \mathcal{S}/\mathbb{C} \times \mathbb{C}^*$. The following lemma implies that the vector field η is non-singular.

LEMMA 1: *The three vectors $\tilde{\eta}_z$, ξ_z and R_z are linearly independent over $\mathbb{C}$ at every point z of $\mathcal{S}$.*

Proof: Suppose that there is a linear relation: $a\tilde{\eta}_z + b\xi_z + cR_z = 0$. For any index j such that $z_j \neq 0$ (i.e. $j \in I_z$) one has

$$a \operatorname{Re}\lambda_j + b\lambda_j + c = 0.$$

Since $z \in \mathcal{S}$ there exist $(d_j)_{j \in I_z}$ with $d_j \geq 0$ and $\sum_{j \in I_z} d_j = 1$ such that $\sum_{j \in I_z} d_j \lambda_j = 0$. From these relations one obtains

$$c = c \cdot (\sum_{j \in I_z} d_j) = 0.$$

Therefore $a\tilde{\eta}_z + b\xi_z = 0$. If $(a, b) \neq (0, 0)$ then the complex numbers $(\lambda_j)_{j \in I_z}$ are on a same real vector line, but this contradicts the condition (2) fulfilled by Λ. Thus $a = b = c = 0$. ∎

Let us construct the 2-form ω. The standard Kählerian form of $\mathbb{C}^n$ is given by $\Omega(x, y) = \operatorname{Im}\langle x, y\rangle = \operatorname{Im}(\sum_{j=1}^{n} x_j \bar{y}_j)$. Denote by Ω_M its restriction to M. Recall that the S^1-action on M is induced by the vector field iR. Observe that $i_{iR}\Omega_M = 0$ and that Ω_M is closed, being a restriction of a closed form. This implies that $L_{iR}\Omega_M = 0$. It follows from these facts that there exists a (unique) closed 2-form ω_1 on M/S^1 such that $\pi^*\omega_1 = \Omega_M$. Now define ω as being equal to $(\phi^{-1})^*\omega_1$ (cf. diagram (D)).

PROPOSITION 2: *The 2-form ω is transversely Kählerian with respect to η.*

Proof: The vector fields U and V on M defined by

$$U = i \cdot \sum_{j=1}^{n} (\operatorname{Re} \lambda_j) z_j \frac{\partial}{\partial z_j} \quad \text{and} \quad V = i \cdot \sum_{j=1}^{n} (\operatorname{Im} \lambda_j) z_j \frac{\partial}{\partial z_j}$$

commute with iR and therefore they project onto vector fields $\pi_* U$ and $\pi_* V$ on M/S^1. One has $\phi_* \pi_* U = \psi_* U = i\eta$ and $\phi_* \pi_* V = \psi_* V = \psi_*(\xi - \tilde{\eta}) = -\eta$. Since

$$\begin{aligned}(TM)_z &= \{\zeta \in \mathbb{C}^n \mid \operatorname{Im}\langle z, \zeta\rangle = \operatorname{Im}\langle U, \zeta\rangle = \operatorname{Im}\langle V, \zeta\rangle = 0\} \\ &= \{\zeta \in \mathbb{C}^n \mid \Omega(iR_z, \zeta) = \Omega(U_z, \zeta) = \Omega(V_z, \zeta) = 0\},\end{aligned}$$

$\operatorname{Ker} \Omega_M$ is generated by iR, U and V. This implies that $\operatorname{Ker} \omega_1$ is generated by $\pi_* U$ and $\pi_* V$. Using the fact that $\omega = (\phi^{-1})^* \omega_1$, one obtains $\operatorname{Ker} \omega = \mathbb{C}\eta$.

Since the 2-form ω is closed by construction it is sufficient to prove that it induces a positive definite Hermitian metric on $TN/\mathbb{C}\eta$. Let us denote by $T_{\mathbb{C}}M$ the maximal complex subbundle of TM, that is $T_{\mathbb{C}}M = TM \cap iTM$. One has

$$(T_{\mathbb{C}}M)_z = \{\zeta \in \mathbb{C}^n \mid \langle z, \zeta\rangle = \langle U, \zeta\rangle = \langle V, \zeta\rangle = 0\}.$$

The bundle $T_{\mathbb{C}}M$ is preserved by the S^1-action induced by iR, being the restriction of a holomorphic action, and projects into a complex bundle $\pi_*(T_{\mathbb{C}}M)$ over M/S^1 included in $T(M/S^1)$. Moreover, given $z \in M$ there is a commutative diagram

$$\begin{array}{ccc} (T_{\mathbb{C}}M)_z & & \\ \downarrow{\scriptstyle \pi_*} & \searrow^{\psi_*} & \\ (\pi_*(T_{\mathbb{C}}M))_{\pi(z)} & \xrightarrow[\phi_*]{} & T(\mathcal{S}/\mathbb{C} \times \mathbb{C}^*)_{\psi(z)} \end{array}$$

where π_* is a $\mathbb{C}$-linear isomorphism and the maps ϕ_* and ψ_* are $\mathbb{C}$-linear injections. The image space $\psi_*((T_{\mathbb{C}}M)_z)$ is naturally identified to the quotient $(TN)_{\psi(z)}/\mathbb{C}\eta$ and the restriction ω' of ω to it is J-invariant (here $J = i$). Therefore the Hermitian form associated to ω' is positive definite since the same is true for the restriction of Ω_M to $(T_{\mathbb{C}}M)_z$. This implies that ω is transversely Kählerian with respect to η. ∎

To each manifold $N = N(\Lambda)$ we associate an integer $k(N)$ defined as being the least positive integer s such that $\mathbf{e}^s = 0$, where $\mathbf{e} \in H^2(M/S^1)$ is the Euler class associated to the S^1-bundle π: $M \to M/S^1$. When N is diffeomorphic to the product of two spheres (the case considered in [2]), then $k(N) = 1$. Otherwise

$1 < k(N) < \dim_{\mathbb{C}} N$ (cf. [3] where a combinatoric definition of $k(N)$ in terms of Λ is also given). Observe now that the Euler class $\mathbf{e}$ is represented in the de Rham group $H^2(M/S^1, \mathbb{R})$ by the 2-form $2\omega_1$. This is verified as follows. The restriction α on M of the 1-form β on $\mathbb{C}^n$ defined by

$$\beta(t) = -\operatorname{Im}\langle R, t\rangle$$

satisfies $\alpha(iR) = 1$ and a direct computation shows $d\beta = 2\Omega$. Therefore $d\alpha = \Omega_M = \pi^*(2\omega_1)$. Finally, one sees that $k(N)$ is the least positive integer s such that ω^s is exact.

Using Proposition 1 one obtains the following theorem:

THEOREM 1: *Every analytic subset of N of pure dimension $\geq k(N)$ is tangent to η. In particular, every complex hypersurface of N is tangent to η.*

In the case where N is diffeomorphic to a product of two spheres, $k(N) = 1$ and therefore any analytic subset of N of positive dimension is tangent to η.

4. Submanifolds of N in the generic case

Let $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ be a sequence of non-zero complex numbers fulfilling conditions (1) and (2) as in Section 2. We say that Λ satisfies the genericity condition (C) if

$$\text{(C)} \qquad \forall c_1, \ldots, c_n \in \mathbb{Q} \quad \text{with} \quad \sum_{j=1}^{p} c_j = 0 \quad \text{one has}$$
$$\sum_{j=1}^{p} c_j \lambda_j = 0 \quad \Rightarrow \quad c_j = 0 \quad \forall j.$$

In any given manifold $N = N(\Lambda)$ there is the following class of complex submanifolds. Let I be a subset of $\{1, 2, \ldots, n\}$ such that $(\lambda_j)_{j\in I}$ is in the Siegel domain and denote

$$\mathcal{S}_I = \{z = (z_1, \ldots, z_n) \in \mathcal{S} \mid z_j = 0 \text{ for } j \notin I\}.$$

Then $\mathcal{S}_I$ is saturated by the action of $\mathbb{C} \times \mathbb{C}^*$ and

$$N_I = \mathcal{S}_I / \mathbb{C} \times \mathbb{C}^*$$

is a closed complex submanifold of N. The following theorem says that, in the generic case, any complex submanifold of a certain dimension is of this type.

THEOREM 2: *Assume that Λ fulfils the genericity condition (C). Then any analytic subset of N of pure dimension $\geq k(N)$ is a union of manifolds N_I.*

Remark: As before, in the case of a product of spheres (cf. [2]), the conclusion holds for any analytic set of positive dimension. In this case, a partial description of the analytic subsets of N, when no genericity assumptions are made, was given in [2].

Proof: The open subset $(\mathbb{C}^*)^n$ of $\mathbb{C}^n$ is contained in S and saturated by the action of $\mathbb{C} \times \mathbb{C}^*$. The quotient

$$G = (\mathbb{C}^*)^n / \mathbb{C} \times \mathbb{C}^*$$

is an open and dense subset of N. It is sufficient to prove that any analytic subset of G, which is tangent to η, is G itself. In fact, an analytic subset Y of N in the hypothesis of the theorem is tangent to η. If Y contains a point of G then $Y \cap G$ is G itself and $Y = N$. Otherwise Y must be contained in some N_I $(I \neq \emptyset)$. But N_I is also a LM-V manifold associated to some $\Lambda_I \subset \Lambda$ and, since Λ satisfies the genericity condition (C), so does Λ_I. Moreover, $k(N_I) \leq k(N)$ as one can see by considering the restriction of ω_1 to N_I. Then one concludes by induction.

Notice that G is a complex Lie group and that the holomorphic vector field η is left invariant. In fact G is isomorphic to the group $(\mathbb{C}^*)^p/\mathbb{C}$, with $p = n - 1$, the isomorphism being induced by the map

$$(z_1, \ldots, z_n) \mapsto \left(w_1 = \frac{z_1}{z_n}, \ldots, w_p = \frac{z_{n-1}}{z_n}\right),$$

and where the $\mathbb{C}$-action on $(\mathbb{C}^*)^p$ is the one associated to the vector field

$$\xi' = \sum_{i=1}^{p} \mu_i w_i \frac{\partial}{\partial w_i} \qquad \text{with } \mu_i = \lambda_i - \lambda_n.$$

Then the holomorphic vector field η is seen as the projection of the vector field $\eta' = \sum_{i=1}^{p} (\mathrm{Re}\, \mu_i) w_i \frac{\partial}{\partial w_i}$. We take from now on this new model for G. Remark that the genericity condition (C) is equivalent to saying that the coefficients $\mu_1, \ldots, \mu_p$ are linearly independent over $\mathbb{Q}$.

Let ϖ denote the canonical projection of $(\mathbb{C}^*)^p$ onto G and set $K = \varpi((S^1)^p)$. Let Y be an analytic subset of G tangent to η. We assume for convenience that the identity element e of G belongs to Y. The statement will follow from the following facts:

(a) the compact subgroup K is the closure of the complex one-parameter group generated by η, and

(b) $(TG)_e = (TK)_e + i \cdot (TK)_e$.

In fact, since Y is tangent to $\mathbb{C}\eta$, condition (a) implies that Y contains K and then $Y = G$ by condition (b).

The Lie algebra $\mathrm{Lie}((\mathbb{C}^*)^p)$ of $(\mathbb{C}^*)^p$ is naturally identified to $\mathbb{C}^p$ in such a way that the exponential map is given by $(w_1, \ldots, w_p) \mapsto (e^{w_1}, \ldots, e^{w_p})$. Then we have $\mathrm{Lie}((S^1)^p) = i\,\mathbb{R}^p$ and $\mathrm{Lie}(G) = \mathbb{C}^p/\mathbb{C} \cdot \boldsymbol{\xi}$, where $\boldsymbol{\xi} = \sum_{i=1}^p \mu_i \vec{e}_i$ and $\vec{e}_1, \ldots, \vec{e}_p$ denotes the canonical basis of $\mathbb{C}^p$. Notice that the Lie algebra morphism ϖ_* coincides with the canonical projection $\mathbb{C}^p \to \mathrm{Lie}(G)$. Now (b) is obvious since $(TK)_e = \varpi_* \mathrm{Lie}((S^1)^p)$ and $\mathrm{Lie}((S^1)^p) + i \cdot \mathrm{Lie}((S^1)^p) = \mathrm{Lie}((\mathbb{C}^*)^p)$.

The vector $\tilde{\boldsymbol{\eta}}$ in $\mathbb{C}^p$ associated to the left invariant vector field $\tilde{\eta}'$ is $\tilde{\boldsymbol{\eta}} = \sum_{i=1}^p (\mathrm{Re}\ \mu_i)\vec{e}_i$. The two vectors $i\tilde{\boldsymbol{\eta}}$ and $\boldsymbol{\xi} - \tilde{\boldsymbol{\eta}}$ belong to $\mathrm{Lie}((S^1)^p)$. This implies that $\mathbb{C}\eta$ is included in $\mathrm{Lie}(K)$. In order to conclude it is sufficient to prove that the subgroup H of $(S^1)^p$ associated to the Lie subalgebra generated by $i\tilde{\boldsymbol{\eta}}$ and $\boldsymbol{\xi} - \tilde{\boldsymbol{\eta}}$ is dense in $(S^1)^p$. For this we show that any continuous H-invariant function f on $(S^1)^p$ is constant. Given $\mathbf{m} \in \mathbb{Z}^p$, let $\hat{f}(\mathbf{m})$ denote the associated Fourier coefficient of f. For any $t \in \mathbb{R}$ one has

$$\hat{f}(\mathbf{m}) \cdot e^{it(\sum(\mathrm{Re}\,\mu_k)m_k)} = \hat{f}(\mathbf{m})$$

and

$$\hat{f}(\mathbf{m}) \cdot e^{it(\sum(\mathrm{Im}\,\mu_k)m_k)} = \hat{f}(\mathbf{m}).$$

This means that, if $\hat{f}(\mathbf{m}) \neq 0$, then

$$\sum(\mathrm{Re}\,\mu_k)m_k = \sum(\mathrm{Im}\,\mu_k)m_k = 0$$

and therefore $\sum m_k\mu_k = 0$. But the $\mathbb{Q}$-independence of the μ_k implies $\mathbf{m} = 0$, showing that f must be constant. This finishes the proof. ∎

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